Current Distribution and Impedance of Lossless Conductor Systems

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Abstract—**A general method for determining the characteristic impedance of uniform, lossless transmission systems is developed. The current distribution within the system is determined by means of a matrix equation programmed for computer solution. Once the current distribution is known, the inductance per unit length and characteristic impedance are determined. The results obtained by applying this method to several rectangular coaxial systems are compared with the predictions of an approximate analytic expression. The reflection coefficient of a variable characteristic impedance coaxial line is measured on a time-domain reflectometer (TDR), and the results are compared with both the matrix method and the approximate analytic expression.**

Introduction

A GENERAL matrix method is developed for determining the current distribution in the transverse plane of any uniform, lossless transmission system. Once the current distribution is known for an arbitrary system (Fig. 1), the inductance per unit length and characteristic impedance can be determined. With an additional step, omitted in this paper, the transverse magnetic field can be mapped for the system under consideration.

This method is not limited to any particular cross-sectional configuration and requires no dimensional restrictions. To provide a comparison with other sources, the method is applied to a rectangular coaxial system, as shown in Fig. 2. This was previously constructed [1] for use as a coaxial reflection standard, but an analytic expression was not available for its characteristic impedance.

When the angle of rotation θ in Fig. 2 is zero or 90°, a simpler parallel configuration is obtained which has been treated by numerous authors in an attempt to obtain an analytic expression for the characteristic impedance. The report by Chen [2] is an example of this approach and his results are used for purposes of comparison. Chen attempts to determine the capacitance per unit length from the configuration of the transverse electric field. To obtain a solution it was necessary to incorporate approximations which in turn require dimensional restrictions on application of the results.

Skiles and Higgins [3] avoid the approximations and restrictions by using ortho-normal block analysis to determine the electric field configuration. The field is then integrated over the inner conductor to produce an expression for the capacitance per unit length. The results are exact but contain infinite series whose convergences are a function of the geometry. Cruzan and Garver [4] refine this approach and adapt it to computer analysis. They also provide a compilation and comparison of approximate solutions which will not be repeated here.

The present method is inherently more direct for obtaining the characteristic impedance of nonanalytic transmission systems since the transverse field need not be obtained first. This provides for a considerable savings of computer storage and time.

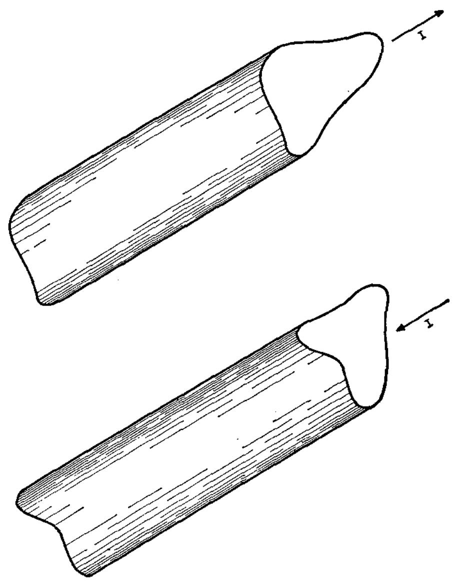

Fig. 1. General self-shielding go-and-return circuit.

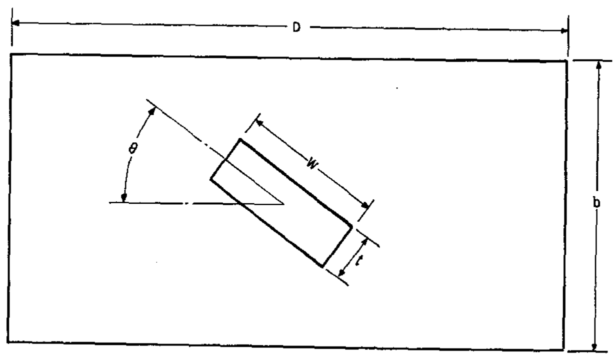

Fig. 2. Cross section of variable impedance line in the nonparallel configuration.

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Theoretical Development

The following development will be restricted to lossless systems supporting a TEM wave. With these restrictions it can be shown [5] that the configuration of the transverse field is exactly a static distribution satisfying Laplace's equation. The electric field boundary conditions are those used for a static field distribution. The magnetic field must be tangential to the conducting surfaces; the magnetic field pattern in the transverse plane corresponds exactly to that arising from a static current distribution confined to the surfaces of the perfect conductors. The differential equations encountered in conventional transmission-line analysis can be derived from Maxwell's equations. This justifies the use of static L's and C's in the following development.

Each of the conductors in Fig. 1 is mathematically subdivided into $4n$ sections with uniform cross sections and parallel longitudinal axes. If the current in each of the $8n$ sections is known, the effective inductance L_{eff} can be obtained directly from the following relation:

$$L_{\text{eff}} = \frac{\sum_{k=1}^{8n} \sum_{l=1}^{8n} I_k \cdot I_l M_{kl}}{|I_{\text{total}}|^2}. \tag{1}$$

Equation (1) is derived from simple energy considerations or Poynting's theorem. Here I_k is the current in section k, M_{kl} is the mutual inductance between sections k and l, and M_{ii} is the self-inductance of section i.

Assuming the current in each subsection to be uniformly distributed throughout its cross section, the coefficients of inductance M_{kl} used in (1) can be obtained from low-frequency relations. Treating each section as an independent conductor, one can write the voltage induced in any single section as

$$v_k = j\omega \sum_{l=1}^{8n} M_{kl} I_l. \tag{2}$$

The entire system of relations is then written in matrix form as

$$\begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ V_{8n} \end{bmatrix} = j\omega \begin{bmatrix} M_{1,1} & \cdots & M_{1,8n} \\ \cdot & & \\ \cdot & & \\ M_{8n,1} & \cdots & M_{8n,8n} \end{bmatrix} \begin{bmatrix} I_1 \\ \cdot \\ \cdot \\ I_{8n} \end{bmatrix} \tag{3}$$

or using a shorter notation

$$V = j\omega MI. \tag{4}$$

Defining section currents to be

$$I_k = -jb_k$$

or

$$I = -jB \tag{5}$$

and combining (4) and (5), one finds that

$$B = \frac{1}{\omega} M^{-1} V. \tag{6}$$

Because the system is lossless, no part of the field penetrates any of the conductor sections. Since the voltage on the conductors is completely arbitrary, it is defined as zero for all sections in one conductor and $1+j0$ for all sections in the other. The voltage matrix is defined as

$$V = \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ V_{8n} \end{bmatrix} = \begin{bmatrix} 0_1 \\ \cdot \\ \cdot \\ 0_{2n} \\ 1_{2n+1} \\ \cdot \\ \cdot \\ 1_{4n} \\ 0_{4n+1} \\ \cdot \\ \cdot \\ 0_{6n} \\ 1_{6n+1} \\ \cdot \\ \cdot \\ 1_{8n} \end{bmatrix}. \tag{7}$$

The indices are chosen to simplify later work. One conductor now has indices $1 \to 2n$ and $4n+1 \to 6n$, while the other has indices $2n+1 \to 4n$ and $6n+1 \to 8n$. In order to use the results of (6) more easily, (1) is transformed into the following relation:

$$L_{\text{eff}} = \frac{1}{\omega \sum_{1}^{2n} b_k + \omega \sum_{4n+1}^{6n} b_k} = \frac{1}{\omega \sum_{2n+1}^{4n} b_k + \omega \sum_{6n+1}^{8n} b_k}. \tag{8}$$

Application

Application of the above method to a real situation where the conductivity is not infinite is justified [5] by the fact that for any efficient transmission line the error will be negligible.

The method of solution outlined above is applied to a coaxial system in which both inner and outer conductors have rectangular cross sections as shown in Fig. 2. The inner conductor will be allowed to rotate about the center to any angle θ. Subsection labeling is shown in Fig. 3.

The outlined method is simple, but the manual performance of the matrix operations is impossibly tedious for all but the lowest order of matrices. It is possible, however, to program a computer to perform the operations and obtain reasonably large matrix orders. This is the approach used in following application.

Since the conductors are lossless, the current will be confined to two infinitesimally thin shells. Only a thin shell on the inner surface of the outer conductor and one on the outer surface of the inner conductor need be considered. The resultant system of two very thin rectangular shells is mathe-

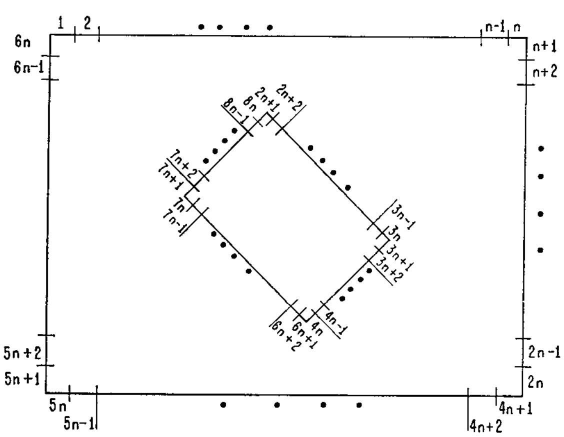

Fig. 3. Section labeling.

matically subdivided into $8n$ very thin tapes with parallel longitudinal axes. The widths of the sections are made small enough to consider the current uniform in each individual section. Each b_k obtained from (6) contains the inverse of ω; consequently, L_{eff} is independent of ω.

The effective inductance can be expressed in matrix notation, using a simple matrix operator defined to be

$$\theta = (0_1 \cdots 0_{2n}, 1_{2n+1} \cdots 1_{4n}, 0_{4n+1} \cdots 0_{6n}, 1_{6n+1} \cdots 1_{8n});$$

the sum of the inner section currents is

$$\sum_{2n+1}^{4n} b_k + \sum_{4n+1}^{8n} b_k = \theta B = \frac{1}{\omega} \theta M^{-1} V \tag{9}$$

and therefore,

$$L_{eff} = 1/\theta M^{-1} V. \tag{10}$$

The independence of inductance with respect to frequency is now apparent. The method outlined here would be exact in the limit as the order of the matrices becomes infinite or as the individual sections become infinitesimal.

Physical symmetry allows a twofold reduction in the order of the matrices. To facilitate this reduction, the section widths W_i chosen so as to maintain physical symmetry are defined to be

$$W_i = W_{4n+i}; \qquad i = 1, 2, 3 \cdots 4n; \tag{11}$$

the symmetry, therefore, continues into the unknown current magnitudes which are represented as

$$b_i = b_{4n+i}; \qquad i = 1, 2, 3 \cdots 4n. \tag{12}$$

The order of (3) can then be reduced to the following form:

$$\begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ V_{4n} \end{bmatrix} = j\omega \left[\begin{bmatrix} M_{1,1} & \cdots & M_{1,4n} \\ \cdot & & \\ \cdot & & \\ M_{4n,1} & \cdots & M_{4n,4n} \end{bmatrix} + \begin{bmatrix} M_{1,4n+1} & \cdots & M_{1,8n} \\ \cdot & & \\ \cdot & & \\ M_{4n,4n+1} & \cdots & M_{4n,8n} \end{bmatrix} \right] \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ b_{4n} \end{bmatrix}. \tag{13}$$

If the current in each section is assumed to be uniform, the M_{kl} in (13) can be calculated from low-frequency relations. Using the relations given in the Appendix, the final program calculates the M_{kl}. It then substitutes these values into (13), inverts to determine the currents, and uses (10) to calculate the effective inductance per unit length. Upon determining the effective inductance, the approximate characteristic impedance is found from the following relation:

$$Z_{eff} = vL_{eff} \tag{14}$$

where v is the velocity of propagation and is assumed to be the velocity of light. The value predicted by the computer for Z_{eff} is a function of the number of subsections and the manner in which the section widths are chosen. The problem of width selection will be considered first.

Method of Subdivision

The method of subdivision given here, although not unique, was found to give good convergence for this particular problem.

Preliminary computer results, with the inner and outer conductors each divided into $4n$ equal parts, indicated that the current distribution is a slowly changing function of subsection position in the outer conductor. The current distribution, however, is a rapidly varying function of section position in the inner conductor where most of the current is concentrated at the corners. For all later solutions the top and side of the outer conductor are each divided into n equal parts.

To validate more closely the assumption that the current is uniform in every individual section, the inner conductor is divided into smaller sections near its corners than at the center. The widths of the first $n/2$ sections in the top of the inner conductor are found from the following arbitrary expression:

$$W_{2n+k} = \frac{C_1}{(n/2 + 2 - k)^\beta}; \; k = 1, 2, \cdots, \frac{n}{2}; \text{ even } n \tag{15}$$

where

$$C_1 = \frac{W}{2} \sum_{k=1}^{n/2} \left[\frac{1}{\left(\frac{n}{2} + 2 - k\right)^\beta} \right]. \tag{16}$$

The value of W is obtained from the geometry as shown in Fig. 2, while β, an arbitrary constant, is varied to produce better convergence. A β of three is found to be satisfactory. For the side of the inner conductor,

$$W_{3n+k} = \frac{C_2}{\left(\frac{n}{2} + 2 - k\right)^\beta}; \qquad k = 1, 2, \cdots, \frac{n}{2}; \quad \text{even } n \tag{17}$$

where

$$C_2 = \frac{t}{2} \sum_{k=1}^{n/2} \left[\frac{1}{\left(\frac{n}{2} + 2 - k\right)^\beta} \right] \tag{18}$$

and the value of t, like W, is determined by the physical dimensions of the problem.

Limiting Value of Impedance

The method of subdivision, although more complex, is similar to that used for a strip line [6]. The latter method [6] used to obtain a limit for the inductance as a function of the matrix order is applicable to this problem. The impedance predicted for finite matrix order can be related to the limiting value ($n \to \infty$) in the following manner:

$$Z_{\text{eff}}(\infty) = Z_{\text{eff}}(n) + an^{-\alpha} \tag{19}$$

where n is the number of sections in one-eighth of the cross section. Choosing four orders of subdivision or matrix order such that

$$n_1/n_2 = n_3/n_4,$$

one finds an expression for the limiting value of inductance in terms of finite results to be

$$Z_{\text{eff}}(\infty) = \frac{Z(n_1)Z(n_4) - Z(n_2)Z(n_3)}{Z(n_1) - Z(n_2) - Z(n_3) + Z(n_4)}. \tag{20}$$

Using order values of 6, 8, 12, and 16 yields convergence of one part in 10^4. This is the uncertainty to be expected in the theoretical results obtained for this paper. The time for each set of four values to be determined on a high-speed digital computer is about $1\frac{1}{2}$ minutes.

An Approximate Expression

The approximate analytic expression derived by Chen [2] for the characteristic impedance of rectangular coaxial lines is

$$Z_c = \frac{94.15}{\frac{W/b}{1 - t/b} + \frac{1}{\pi}\left\{\left(\frac{1}{1 - t/b}\right) \ln \left(\frac{2 - t/b}{t/b}\right) + \ln \left[\frac{t/b(2 - t/b)}{(1 - t/b)^2}\right]\right\}}. \tag{21}$$

This expression is valid only when the angle between the conductors in Fig. 2 is zero or 90°. A further dimensional limitation requires $t \leq b/2$. The use of (21) in the 90° configuration violates the dimensional restriction imposed on it by Chen; nevertheless, good agreement is still obtained between it and the matrix solution values.

The following empirical expression obtained from reference [1] is used to extend the application of (21) to the rotated case:

$$Z(\theta) = \tfrac{1}{2}[Z_c(0)(\cos 2\theta + 1) + Z_c(90)(1 - \cos 2\theta)]. \tag{22}$$

Values for characteristic impedance obtained for three sets of dimensions from the computer program, Z_l, and (21), Z_c, are shown in Table I. The agreement is very good, especially where the limitation to Chen's relation has not been violated. The computer solution contains no dimensional limitation and is, therefore, generally applicable.

TABLE I
Comparison of Calculated Results

1	2	3	4	5
W/b	t/b	θ (deg)	Z_c (ohms)	Z_l (ohms)
0.6499	0.2592	0	57.66	57.69
0.6499	0.2592	90	46.37	46.28
0.6429	0.2945	0	55.15	55.19
0.6429	0.2945	90	44.79	44.64
0.6429	0.3083	0	54.12	54.12
0.6429	0.3083	90	43.98	43.83

W, t, b, and θ = line parameters (see Fig. 2).
Z_c = approximate solution.
Z_l = matrix solution.

Experimental Setup

A rectangular transmission line with a rotating outer and fixed inner conductor is described in reference [1]. A time-domain reflectometer (TDR) [7]–[10] and x-y recording system were used to plot the characteristic impedance Z of the rectangular line as a function of angle of rotation. Precision coaxial air lines with various characteristic impedances were used to calibrate the TDR and recording system as outlined in reference [1]. The step discontinuities at the ends of the rectangular line have largely been compensated for experimentally.

Conclusion

Figures 4, 5, and 6 show the experimental curves of the three cases discussed in this paper. The points superimposed on the curves are the calculated matrix values (circles) and the approximate values obtained with (22) (solid points). The outer conductor in every case has a width D and a height b of 3.500 and 0.750 inches, respectively.

Examination of Figs. 4, 5, and 6 shows that maximum impedance deviations occur in the zero and 90° positions of the variable impedance line. The maximum deviation between measured and matrix predicted characteristic impedance is, in all cases, less than 0.4 percent. Between 20° and 70° the difference is less than 0.1 percent.

The solution of (22) and the matrix solution agree so closely with each other from 0 to 45° that they are practically indistinguishable on the curves. Only where the dimensional limitations of Chen are violated is the difference observed.

The results verify the applicability of the inductance matrix approach to the rectangular transmission line. The results show that a programmed matrix solution as outlined in this paper should be feasible for any geometrical configuration which can support a TEM mode. The accuracy of the results will be limited only by storage capacity of the computer.

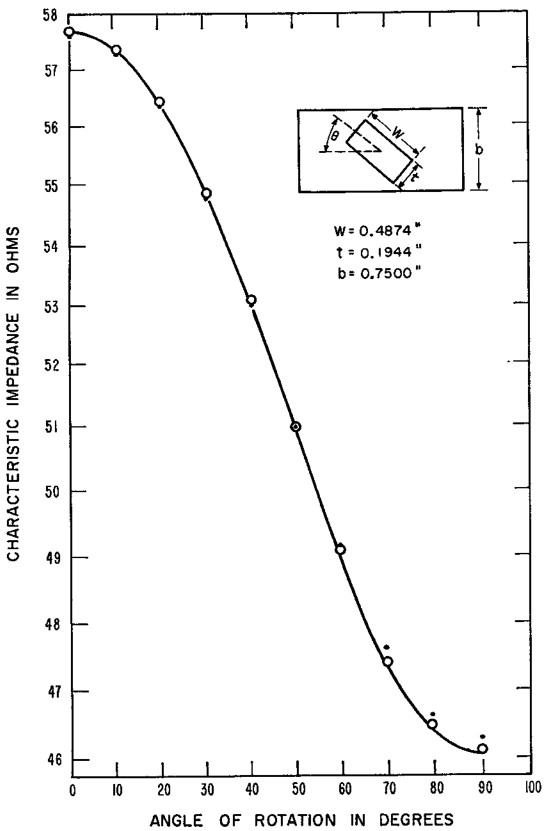

Fig. 4.

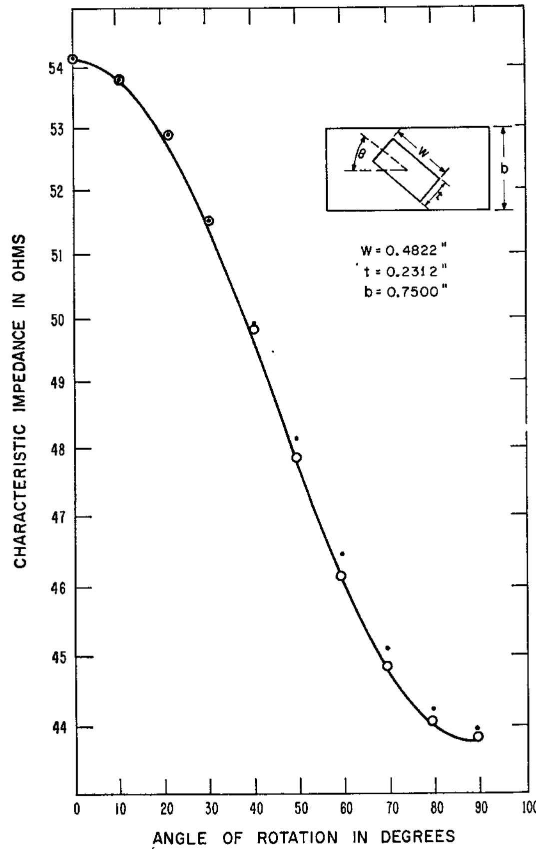

Fig. 5.

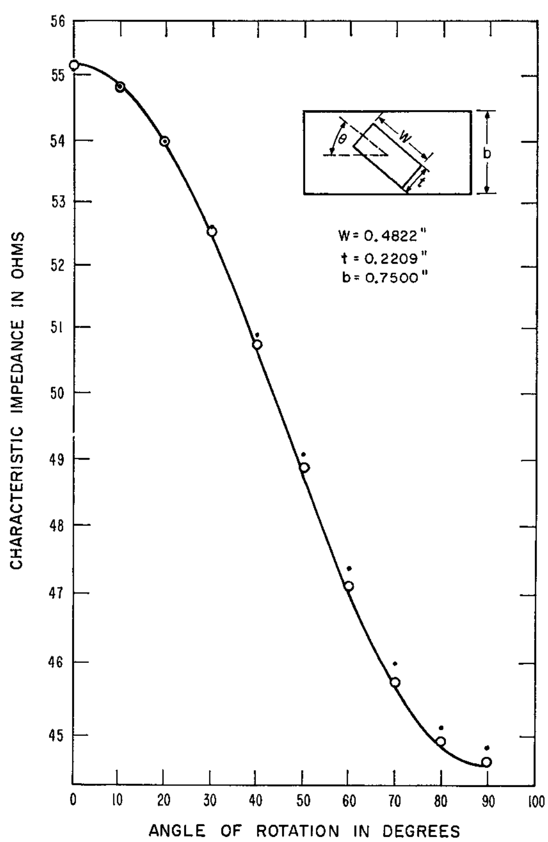

Fig. 6.

Figs. 4, 5 and 6. Characteristic impedance versus angle of rotation, with theoretical matrix values (circles) and approximate analytic values (points) superimposed on the experimental curves.

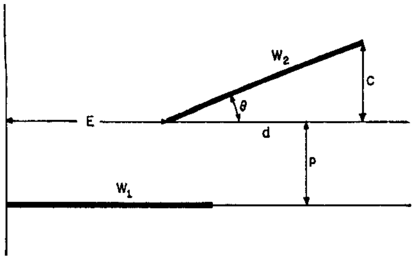

Fig. 7. Cross section of nonparallel tapes.

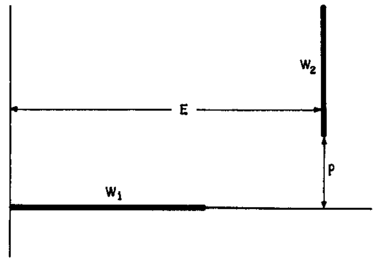

Fig. 8. Cross section of perpendicular tapes.

$$\begin{aligned} m/l = \frac{0.2}{\omega_1\omega_2 \cos\theta} \Bigg\{ & \frac{1}{4}\left[(x_2 - x_1)^2 - \frac{(mx_1 + b)^2}{m^2+1}\right] \ln\left[(x_2 - x_1)^2 + (mx_2 + b)^2\right] \\ & + \frac{1}{2}\left[\frac{mb^2 - mx_1^2 - 2bx_1}{m^2+1}\right] \tan^{-1}\left[\frac{(m^2+1)x_2 + (mb - x_1)}{mx_1 + b}\right] \\ & + x_2\left(\frac{1}{2}mx_2 + b\right) \tan^{-1}\left[\frac{x_2 - x_1}{mx_2 + b}\right]\Bigg\} \Bigg|_0^{\omega_1} (x_1) \Bigg|_E^{E+\omega_1 \cos\theta} (x_2) \\ & + 0.2\left[\ln\left(2l + \frac{1}{2}\right)\right] \end{aligned} \tag{24}$$

This method provides an effective technique for determining inductance and current distribution of transmission-line configurations for which an analytic solution is too difficult to obtain.

Summary

A matrix method has been developed for determining the inductance and characteristic impedance of a rectangular transmission line. This line may have any dimensions, and the coaxial conductors do not have to be in a parallel cross-sectional configuration.

The results of the matrix approach have been compared with values obtained from Chen's equation. The matrix solution is also compared with the experimental results obtained with a TDR system. These results show the maximum measured characteristic impedance deviation to be less than 0.4 percent from the calculated values.

Appendix

The application of the method outlined in this paper requires the use of dc relations for the determination of the inductance coefficients. Application of Neumann's form for the mutual inductance equation to the geometries in question is straightforward although somewhat tedious.

The self-inductance per unit length of a thin tape, whose length l is much greater than its width w, is found to be

$$\mathcal{L}/l = 0.2[\ln 2l - \ln W + \tfrac{1}{2}]. \tag{23}$$

The mutual inductance per unit length between two very long thin tapes with parallel longitudinal axes but nonparallel transverse axes is far more tedious to obtain. The geometry of the transverse plane is shown in Fig. 7. The final results are found to be

where $m = \tan\theta$, and $b = p - E\tan\theta$.

For compactness as well as ease of programming, the limits on the integration of the two variables x_1 and x_2 have been included in the final results. This relation is adequate for any angle of rotation, except $\pi/2$ or an odd multiple. It was therefore necessary to determine the mutual coefficient for this angle as a special case.

The mutual inductance per unit length of two long thin tapes with parallel longitudinal axes and perpendicular transverse axes as shown in Fig. 8 is found to be

$$\begin{aligned} m/l = \frac{0.2}{\omega_1\omega_2} \Bigg\{ & -x_1x_2 \ln\left[x_1^2 + x_2^2\right]^{1/2} \\ & + \frac{x_1^2 - x_2^2}{2} \tan^{-1}\left(\frac{x_1}{x_2}\right)\Bigg\} \Bigg|_{E-\omega_1}^{E} (x_1) \Bigg|_P^{P+\omega_2} (x_2) \\ & + 0.2[\ln(2l) + \tfrac{1}{2}] \end{aligned} \tag{25}$$

where the integration limits on x_1 and x_2 have again been retained in the final results.

It is noted that the results obtained above are not length-independent. The reason for this is the application to individual circuit segments rather than a closed system. For a go-and-return system traversed by currents in opposite directions all such terms will cancel, producing effective inductance coefficients which are independent of length. The term $[\ln(2l) + \frac{1}{2}]$ can therefore be deleted for the go-and-return system.

References

[1] J. E. Cruz and R. L. Brooke, "A variable characteristic impedance coaxial line," *IEEE Trans. on Microwave Theory and Techniques* (*Correspondence*), vol. MTT-13, pp. 477–478, July 1965.
[2] T. S. Chen, "Determination of the capacitance, inductance, and characteristic impedance of rectangular lines," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-8, pp. 510–519, September 1960.
[3] J. J. Skiles and T. J. Higgins, "Determination of the characteristic impedance of UHF coaxial rectangular transmission lines," *Proc. Nat'l Electronics Conf.* (Chicago, Ill.), vol. 10, pp. 97–108, October 1954.
[4] O. R. Cruzan and R. V. Garver, "Characteristic impedance of rectangular coaxial transmission lines," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 488–495, September 1964.
[5] S. Ramo, J. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*. New York: Wiley, 1965 (general analysis of guided waves, pp. 393–400).
[6] R. L. Brooke, C. A. Hoer, and C. H. Love, "Inductance and characteristic impedance of strip-lines," *J. Research NBS* (*Engineering and Instrumentation*), vol. 71C, pp. 59–67, January–March 1967.
[7] R. Carlson, S. Krakauer, R. Monnier, V. Van Duzer, and R. Woodbury, "Sampling oscillography," Hewlett-Packard Applications Note 36, November 1959.
[8] "Transmission line testing using the sample oscilloscope," Hewlett-Packard Applications Note 53.
[9] "Time-domain reflectometry," Hewlett-Packard Applications Note 62, 1964.
[10] "Pulse reflection measurement of transmission line impedance and discontinuities," in-house publication, Tektronix, 1962.

Radiation from an Infinite Array of Parallel-Plate Waveguides with Thick Walls

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Abstract—**A semi-infinite array of parallel-plate waveguides with walls of finite thickness is excited by incident TEM modes in every waveguide identically. By proper application of the boundary conditions, two Wiener-Hopf equations are obtained which, however, cannot be solved by the standard techniques. A method originated by Jones [6] is applied to recast these two equations so that the forms of the solutions are found. The solutions involve constants to be determined by an infinite set of linear simultaneous equations which converge absolutely. When the thickness of the walls b is small compared with the wavelength λ, explicit solutions in the order of $0(b/\lambda)$ are found in very simple forms.**

I. Introduction

THE PROBLEM of radiation from an infinite array of parallel-plate waveguides is of great interest theoretically and practically. In the theoretical aspect, it offers an excellent example for the study of periodic structures. In particular, it was one of the first problems solved exactly by the Wiener-Hopf technique [1]. From the practical point of view, it simulates a phased array of waveguides which is widely used in today's communication and radar systems. Wu and Galindo [2], for example, made an interesting investigation of the mutual coupling effects of phased arrays by using the solution of this problem.

Most of the analyses in connection with this problem are based on the assumption that the walls of the guides are vanishingly thin. In practice, however, walls of appreciable thickness are unavoidable. Therefore, it is desirable to study the effect of this thickness on the radiation properties. Among past works on the thick-wall probem, Epstein [3] gave an empirical correction to the case of infinitely thin wall based on experimental evidence. After an unsuccessful attempt to find a rigorous theoretical solution, Primich [4] attacked the problem by variational techniques, and obtained some results checked well by experiments. Most recently, Galindo and Wu [5] formulated the problem as an integral equation which is valid for *all* scanning angles. However, that integral equation, as stated by the authors, is nonintegrable, and numerical methods using a high-speed computer were resorted to for an approximated solution.

It is the purpose of this paper to present a solution based on the Wiener-Hopf technique for the broadside radiation of an infinite array of parallel-plate waveguides with thick walls. Particularly when the thickness of the wall is small in terms of wavelength, very simple expressions for the reflection coefficient and the radiated far field are obtained. Because of the complications and the lengthiness of the manipulations, some detailed derivations are omitted in this paper; interested readers are referred to a technical report under the same title issued by Hughes Aircraft Company [10].

II. Statement of the Problem

Consider an infinite array of parallel-plate waveguides as shown in Fig. 1. The thickness of the guide wall is b, and the width of the guide is a. The dominant TEM modes are excited inside every waveguide with equal amplitude and phase. The problem is then to find the radiated field in the

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